# Madras College Maths Department
# Higher Maths
# E&F 1.4 Vectors



Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

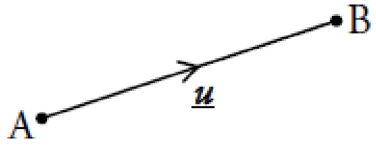
## Vectors and Scalars

A **scalar** is a quantity with magnitude (size) only – for example, an amount of money or a length of time.

Sometimes size alone is not enough to describe a quantity – for example, directions to the nearest shop. For this we need to know a magnitude (i.e. how far), *and* a direction.

Quantities with magnitude and direction are called **vectors**.

A vector is named either by using the letters at the end of a directed line segment (e.g. $\overrightarrow{AB}$ represents a vector starting at point A and ending at point B) or by using a bold letter (e.g. ***u***). You will see bold letters used in printed text, but in handwriting you should just underline the letter.



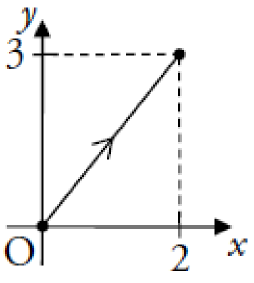

Throughout these notes, we will show vectors in bold as well us underlining them (e.g. $\underline{\boldsymbol{u}}$).

## Components

A vector may be represented by its **components**, which we write in a column. For example,

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is a vector in two dimensions.}$$

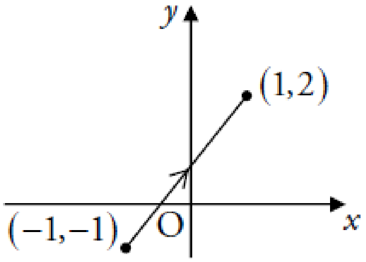
In this case, the first component is 2 and this tells us to move 2 units in the $x$-direction. The second component tells us to move 3 units in the $y$-direction. So if the vector starts at the origin, it will look like:





Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

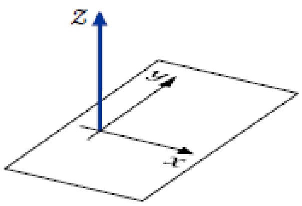
Note that we write the components in a column to avoid confusing them with coordinates. The following diagram also shows the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, but in this case it does not start at the origin.



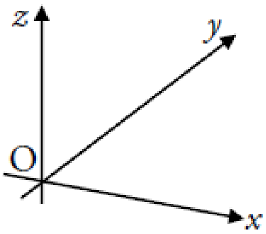

## Vectors in Three Dimensions

In a vector with three components, the first two tell us ho many units to move in the *x*- and *y*-directions, as before. The third component specifies how far to move in the *z*-direction.

When looking at a pair of $(x, y)$ -axes, the *z*-axis points out of the page from the origin.



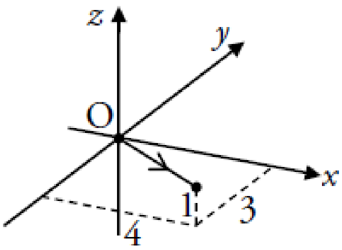

A set of 3D axes can be drawn on a page as shown to the right.




For example,

$$\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

is a vector in three dimensions. This vector is shown in the diagram, starting from the origin.




## Zero Vectors

Any vector with all components zero is called a zero vector and can be written as $\underline{\mathbf{0}}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{\mathbf{0}}$.

## Magnitude

The **magnitude** (or length) of a vector $\underline{\boldsymbol{u}}$ is written as $|\underline{\boldsymbol{u}}|$. It can be calculated as follows.

$$\text{If } \boldsymbol{u} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ then } |\underline{\boldsymbol{u}}| = \sqrt{a^2 + b^2}$$

$$\text{If } \boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ then } |\underline{\boldsymbol{u}}| = \sqrt{a^2 + b^2 + c^2}$$

**EXAMPLES**

1. Given $\underline{\boldsymbol{u}} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, find $|\underline{\boldsymbol{u}}|$.

2. Find the length of $\underline{\boldsymbol{a}} = \begin{pmatrix} -\sqrt{5} \\ 6 \\ 3 \end{pmatrix}$.

### Unit Vectors

Any vector with a magnitude of one is called a **unit vector**. For example:

$$\text{if } \underline{\boldsymbol{u}} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \text{ then } |\underline{\boldsymbol{u}}| = \sqrt{\left(\tfrac{1}{2}\right)^2 + 0^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\tfrac{4}{4}}$$

$$= 1 \text{ unit.}$$

So $\underline{\boldsymbol{u}}$ is a unit vector.



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The vector of unit magnitude in the direction of $\underline{a}$ is given by $\hat{\underline{a}}$

where $$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

The following relation by rearrangement is very important

$$\hat{\underline{a}} \times |\underline{a}| = \underline{a}$$

Direction only

Magnitude only

Q1) Calculate the unit vector in the direction of $\underline{u} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$

Q2) Calculate the unit vector in the direction of $\underline{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

### Distance in Three Dimensions

The distance between the points A and B is $d_{\mathrm{AB}} = \left|\overrightarrow{\mathrm{AB}}\right|$ units.

For example, given $\overrightarrow{\mathrm{AB}} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, we find $d_{\mathrm{AB}} = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{30}$.

In fact, there is a three-dimensional version of the distance formula.

The distance $d$ between the points $(x_1, y_1, z_1)$ and $(x_1, y_1, z_1)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \text{ units.}$$

**EXAMPLE**

Find the distance between the points $(-1,4,1)$ and $(0,5,-7)$.

## Equal Vectors

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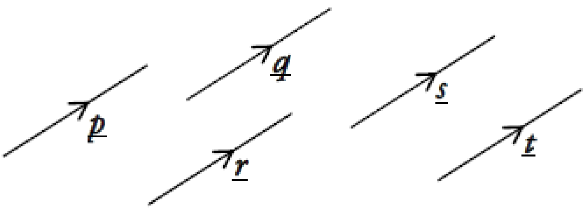
Vectors with the same magnitude and direction are equal.

For example, all the vectors shown to the right are equal.



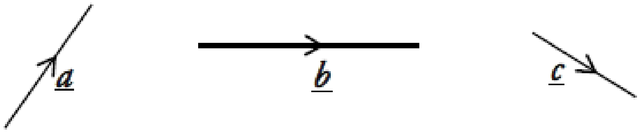

If vectors are equal to each other, then all of their components are equal, i.e.

$$\text{if } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ then } a = d,\ b = e \text{ and } c = f.$$

Conversely, two vectors are only equal if all of their components are equal.

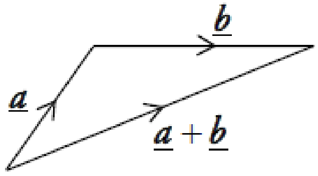
## Addition and Subtraction of Vectors

Consider the following vectors:



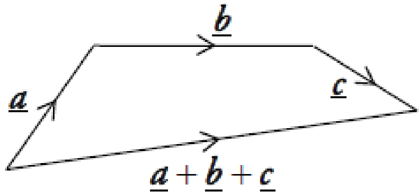

### Addition

We can construct $\underline{a}+\underline{b}$ as follows:




$\underline{a}+\underline{b}$ means $\underline{a}$ followed by $\underline{b}$.

Similarly, we can construct $\underline{a}+\underline{b}+\underline{c}$ as follows:

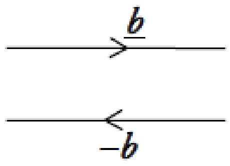



$\underline{a}+\underline{b}+\underline{c}$ means $\underline{a}$ followed by $\underline{b}$ followed by $\underline{c}$.

To add vectors, we position them nose-to-tail. Then the sum of the vectors is the vector between the first tail and the last nose.

### Subtraction

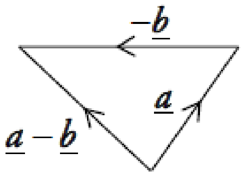
Now consider $\underline{a}-\underline{b}$. This can be written as $\underline{a}+(-\underline{b})$, so if we first find $-\underline{b}$, we can use vector addition to obtain $\underline{a}-\underline{b}$.




$-\underline{b}$ is just $\underline{b}$ but in the opposite direction.

$-\underline{b}$ and $\underline{b}$ have the same magnitude, i.e. $|\underline{b}|=|-\underline{b}|$.

Therefore we can construct $\underline{a}-\underline{b}$ as follows:




$\underline{a}-\underline{b}$ means $\underline{a}$ followed by $-\underline{b}$.

## Using Components

If we have the components of vectors, then things become much simpler.

The following rules can be used for addition and subtraction.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix} \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$$

add the components subtract the components

**EXAMPLES**

1. Given $\underline{\boldsymbol{u}} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ and $\underline{\boldsymbol{v}} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, calculate $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}}$.

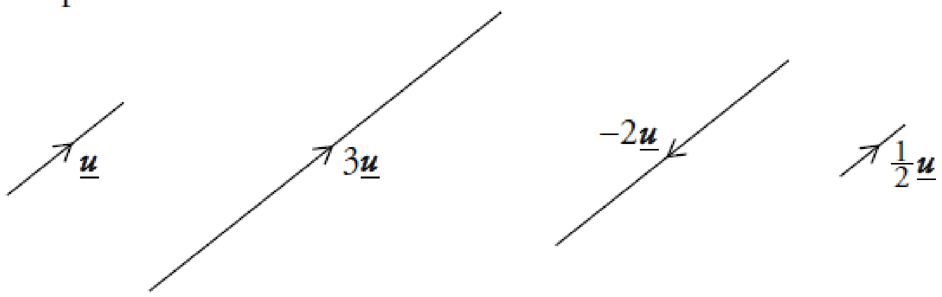
2. Given $\underline{\boldsymbol{p}} = \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}$ and $\underline{\boldsymbol{q}} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}$, calculate $\underline{\boldsymbol{p}} - \underline{\boldsymbol{q}}$ and $\underline{\boldsymbol{q}} + \underline{\boldsymbol{p}}$.

## Multiplication by a Scalar

A vector $\underline{u}$ which is multiplied by a scalar $k > 0$ will give the result $k\underline{u}$. This vector will be $k$ times as long, i.e. the magnitude will be $k|\underline{u}|$.

Note that if $k < 0$ this means that the vector $k\underline{u}$ will be in the opposite direction to $\underline{u}$.

For example:




$$\text{If} \quad \underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{then} \quad k\underline{u} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}.$$

Each component is multiplied by the scalar.

**EXAMPLES**

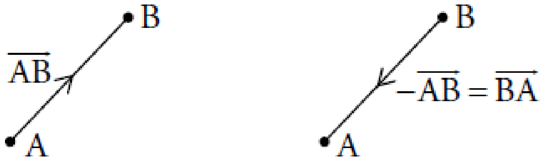
1. Given $\underline{v} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$, find $3\underline{v}$.

2. Given $\underline{r} = \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}$, find $-4\underline{r}$.

### Negative Vectors

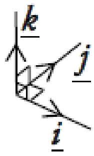
The negative of a vector is the vector multiplied by $-1$.

If we write a vector as a directed line segment $\overrightarrow{AB}$, then $-\overrightarrow{AB} = \overrightarrow{BA}$:




## Basis Vectors

A vector may also be defined in terms of the basis vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$. These are three mutually perpendicular unit vectors (i.e. they are perpendicular to each other).




These basis vectors can be written in component form as

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector can be written in **basis form** using $\underline{i}$, $\underline{j}$ and $\underline{k}$. For example:

$$\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\underline{i} - 3\underline{j} + 6\underline{k}.$$

There is no need for the working above if the following is used:

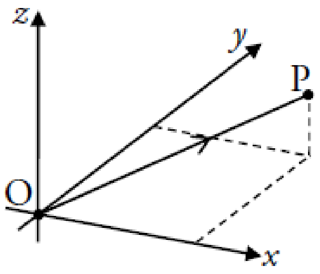
$$a\underline{i} + b\underline{j} + c\underline{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

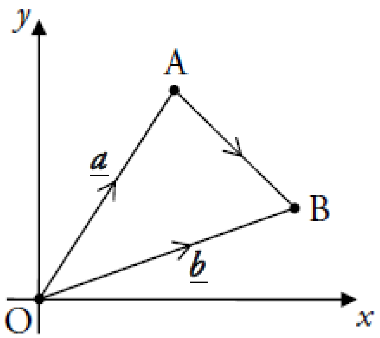
## Position Vectors

$\overrightarrow{OA}$ is called a **position vector** of point A relative to the origin O, and is written as $\underset{\sim}{a}$.

$\overrightarrow{OB}$ is called the position vector of point B, written $\underset{\sim}{b}$.

Given $P(x, y, z)$, the position vector $\overrightarrow{OP}$ or $\underset{\sim}{p}$ has components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.




To move from point A to point B we can move back along the vector $\underset{\sim}{a}$ to the origin, and along vector $\underset{\sim}{b}$ to point B.

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -\underset{\sim}{a} + \underset{\sim}{b} \\ &= \underset{\sim}{b} - \underset{\sim}{a} \end{aligned}$$

For the vector joining any two points P and Q, $\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$.

Question 1 R is the point $(2, -2, 3)$ and S is the point $(4, 6, -1)$. Find $\overrightarrow{RS}$.

Question 2 PQRS is a parallelogram. If P is point (1, 6, 9), Q is (9, 7, 13) and R is (10, 9, 12) calculate the coordinates of S.

## Dividing Lines in a Ratio

There is a simple process for finding the coordinates of a point which divides a line segment in a given ratio.

**EXAMPLE**

1. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

   The point T divides PR in the ratio $2:3$. Find the coordinates of T.

**OR**

### Using the Section Formula

The previous method can be condensed into a formula as shown below.

If the point P divides the line AB in the ratio $m:n$, then:

$$\underline{p} = \frac{n\underline{a} + m\underline{b}}{n+m},$$

where is $\underline{a}$, $\underline{b}$ and $\underline{p}$ are the position vectors of A, B and P respectively.

It is not necessary to know this, since the approach explained above will always work.

**EXAMPLE**

2. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

   The point T divides PR in the ratio $2:3$. Find the coordinates of T.

## Vector Pathways

We can use the rules of adding and taking away vectors to express a vector $\overrightarrow{AB}$ in a geometrical situation as a combination of other, known, vectors.

To do this, we identify a route, or pathway, between A and B, in which each step of the route can be expressed in terms of one of the other known pathways. We can choose any route we like, and the final answer, when simplified, will always be the same.

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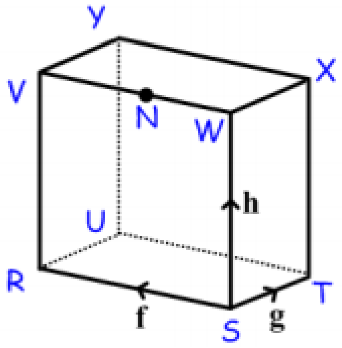
Example – vector pathways (adapted from 2008 SQA exam question)

**In the diagram, RSTUVWXY represents a cuboid.**

**$\overrightarrow{SR}$ represents vector $\underline{\mathbf{f}}$, $\overrightarrow{ST}$ represents vector $\underline{\mathbf{g}}$ and $\overrightarrow{SW}$ represents vector $\underline{\mathbf{h}}$.**

**N is the midpoint of VW.**

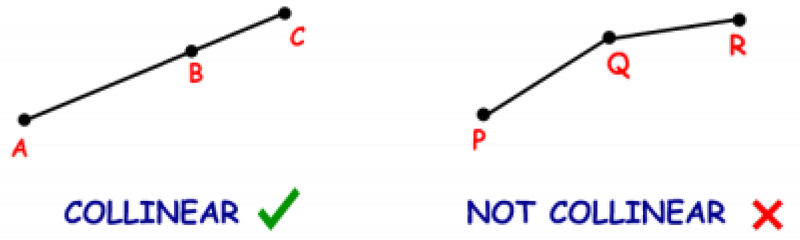
**Express $\overrightarrow{RX}$ and $\overrightarrow{TN}$ in terms of $\underline{\mathbf{f}}$, $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$.**




**Solution**

## Collinearity

Three or more points are said to be **collinear** if they lie on the same straight line:




The points A, B and C in 3D space are collinear if $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, with B a common point.

Note that we *cannot* find gradients in three dimensions – instead we use the following.

Non-zero vectors are **parallel** if they are scalar multiples of the same vector.

For example:

If $\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 3\underline{u}$ then $\underline{u}$ and $\underline{v}$ are parallel.

If $\underline{p} = \begin{pmatrix} 15 \\ 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 20 \\ 12 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ then $\underline{p}$ and $\underline{q}$ are parallel.

**EXAMPLE**

A is the point $(1, -2, 5)$, $B(8, -5, 9)$ and $C(22, -11, 17)$.

Show that A, B and C are collinear.

Example 2 – Prove that A(3, 4, 1), B(9, 1, -5) and C(11, 0, -7) are collinear

## Zero Vectors

Any vector with all components zero is called a zero vector and can be written as $\underline{\mathbf{0}}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{\mathbf{0}}$.

Question – 3 Forces, $F_A$, $F_B$ and $F_C$, acting on an object are in equilibrium.

Given that $F_A=5\underline{\mathbf{i}} + 6\underline{\mathbf{j}} + a\underline{\mathbf{k}}$, $F_B=b\underline{\mathbf{i}} - 3\underline{\mathbf{j}} - 7\underline{\mathbf{k}}$ and $F_C= -2\underline{\mathbf{i}} + c\underline{\mathbf{j}} - 9\underline{\mathbf{k}}$ calculate the values of a, b and c.



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## The Scalar Product

So far we have added and subtracted vectors and multiplied a vector by a scalar. Now we will consider the scalar product, which is a form of vector multiplication.

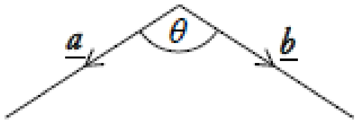
The **scalar product** is denoted by $\underline{a}.\underline{b}$ (sometimes it is called the dot product) and can be calculated using the formula:

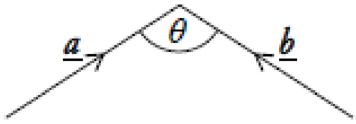
$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta\,,$$

where $\theta$ is the angle between the two vectors $\underline{a}$ and $\underline{b}$.

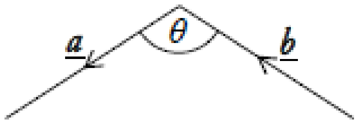
This formula is given in the exam.

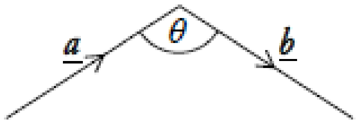
The definition above assumes that the vectors $\underline{a}$ and $\underline{b}$ are positioned so that they both point away from the angle, or both point into the angle.




However, if one vector is pointing away from the angle, while the other points into the angle,

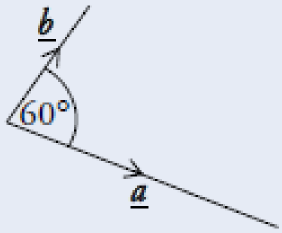



we find that $\underline{a}.\underline{b} = -|\underline{a}||\underline{b}|\cos\theta$.

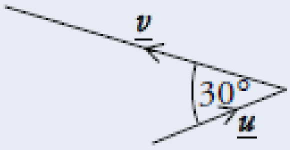
**EXAMPLES**

1. Two vectors, $\underline{a}$ and $\underline{b}$ have magnitudes 7 and 3 units respectively and are at an angle of $60°$ to each other as shown below.




What is the value of $\underline{a}.\underline{b}$?

2. The vector $\underline{u}$ has magnitude $k$ and $\underline{v}$ is twice as long as $\underline{u}$. The angle between $\underline{u}$ and $\underline{v}$ is $30°$, as shown below.




Find an expression for $\underline{u}.\underline{v}$ in terms of $k$.

## The Component Form of the Scalar Product

The scalar product can also be calculated as follows:

$$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 \quad \text{where } \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

This is given in the exam.

**EXAMPLES**

3. Find $\underline{p}.\underline{q}$, given that $\underline{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

4. If A is the point $(2, 3, 9)$, $B(1, 4, -2)$ and $C(-1, 3, -6)$, calculate $\overrightarrow{AB}.\overrightarrow{AC}$.



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## The Angle Between Vectors

The formulae for the scalar product can be rearranged to give the following equations, both of which can be used to calculate $\theta$, the angle between two vectors.

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} \qquad \text{or} \qquad \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\underline{a}||\underline{b}|}.$$

Look back to the formulae for finding the scalar product, given on the previous pages. Notice that the first equation is simply a rearranged form of the one which can be used to find the scalar product. Also notice that the second simply replaces $\underline{a}.\underline{b}$ with the component form of the scalar product.

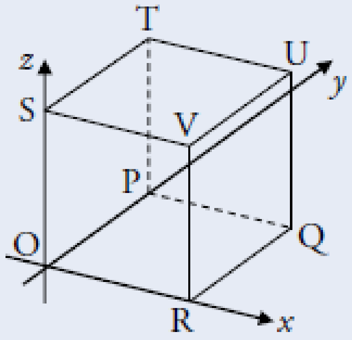
These formulae are *not* given in the exam but can both be easily derived from the formulae on the previous pages (which *are* given in the exam).

**EXAMPLES**

1. Calculate the angle $\theta$ between vectors $\underline{p} = 3\underline{i} + 4\underline{j} - 2\underline{k}$ and $\underline{q} = 4\underline{i} + \underline{j} + 3\underline{k}$.

2. K is the point $(1, -7, 2)$, $L(-3, 3, 4)$ and $M(2, 5, 1)$. Find $K\hat{L}M$.

3. The diagram below shows the cube OPQRSTUV.




The point R has coordinates $(4,0,0)$.

(a) Write down the coordinates of T and U.

(b) Find the components of $\overrightarrow{RT}$ and $\overrightarrow{RU}$.

(c) Calculate the size of angle TRU.

## Perpendicular Vectors

If $\underline{a}$ and $\underline{b}$ are perpendicular then $\underline{a}.\underline{b} = 0$.

This is because $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

$$= |\underline{a}||\underline{b}|\cos 90° \quad (\theta = 90° \text{ since perpendicular})$$

$$= 0 \quad (\text{since } \cos 90° = 0)$$

Conversely, if $\underline{a}.\underline{b} = 0$ then $\underline{a}$ and $\underline{b}$ are perpendicular.

**EXAMPLES**

1. Two vectors are defined as $\underline{a} = 4\underline{i} + 2\underline{j} - 5\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 2\underline{k}$.
   Show that $\underline{a}$ and $\underline{b}$ are perpendicular.

2. $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ a \\ 7 \end{pmatrix}$ and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \\ a \end{pmatrix}$ where $a$ is a constant.

   Given that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are perpendicular, find the value of $a$.

## Properties of the Scalar Product

Some properties of the scalar product are as follows:

$$\underline{a}.\underline{b} = \underline{b}.\underline{a}$$

$$\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c} \qquad \text{(Expanding brackets)}$$

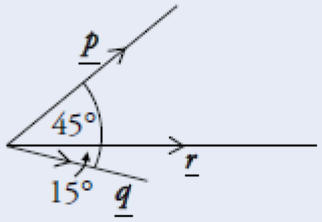
$$\underline{a}.\underline{a} = |\underline{a}|^2$$

Note that these are not given in the exam.

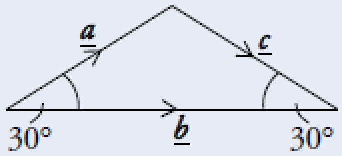
**EXAMPLES**

1. In the diagram, $|\underline{p}| = 3$, $|\underline{r}| = 4$ and $|\underline{q}| = 2$.

   Calculate $\underline{p}.(\underline{q} + \underline{r})$.




2. In the diagram below $|\underline{a}| = |\underline{c}| = 2$ and $|\underline{b}| = 2\sqrt{3}$.




   Calculate $\underline{a}.(\underline{a} + \underline{b} + \underline{c})$.



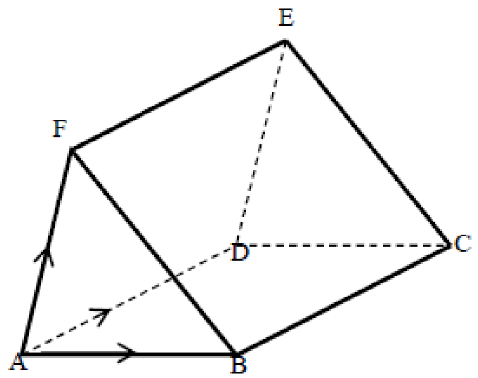
Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

# Unit Assessment Practice (1)

1 ABCDEF is a triangular prism as shown.

The vectors $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AF}$ are given by:

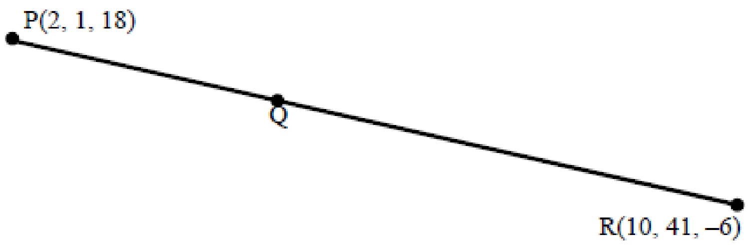
$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 8 \\ 4 \end{pmatrix}$; $\overrightarrow{AD} = \begin{pmatrix} 10 \\ 4 \\ 2 \end{pmatrix}$; $\overrightarrow{AF} = \begin{pmatrix} -1 \\ -4 \\ 13 \end{pmatrix}$




Express $\overrightarrow{FB}$ in component form. 3

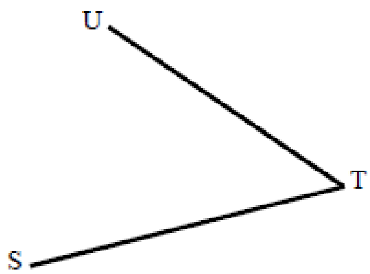
2 Show that the points A(3, -2, 5), B(7, -4, 9) and C(15, -8, 17) are collinear 3

3 The points P, Q and R lie in a straight line, as shown. Q divides PR in the ratio 3:5.




Find the coordinates of Q. 3

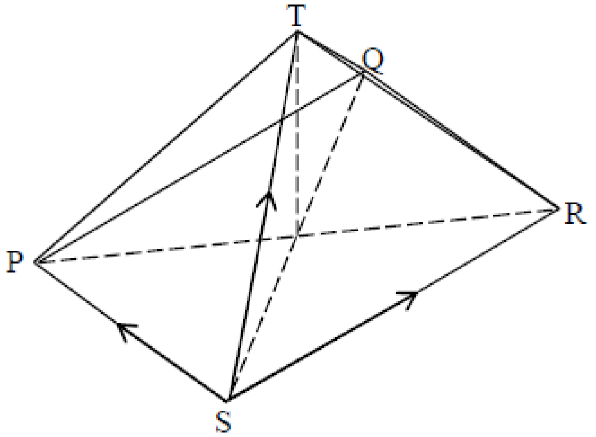
4 Points S, T and U have coordinates S(3, 0, 2), T(7, 1, -5) and U(4, 3, -2).




Find the size of the acute angle STU. 5

# Unit Assessment Practice (2)

1 TPQRS is a pyramid with rectangular base PQRS.




The vectors $\overrightarrow{SP}, \overrightarrow{SR}, \overrightarrow{ST}$ are given by:

$$\overrightarrow{SP} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{SR} = 12\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\overrightarrow{ST} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$$

Express $\overrightarrow{PT}$ in component form.

3

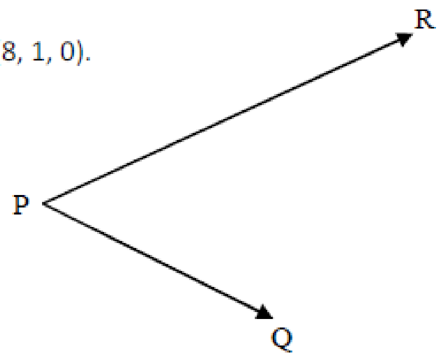
2 Are that the points A(-9, 4, -7), B(-4, 3, -1) and C(11, 0, 11) collinear? 3

3 The points R, S and T lie in a straight line, as shown. S divides RT in the ratio 3:5.

Find the coordinates of S.


3

4 Points P, Q and R have coordinates P(5, −3, −1), Q(7, −4, 2) and R(8, 1, 0).




Find the size of the acute angle QPR.

5

## Homework - Non-Calculator Section

Three vectors can be expressed as follows:

$\overrightarrow{FG} = -2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{GH} = 3\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$

$\overrightarrow{EH} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

(a) Find $\overrightarrow{FH}$. **2**

(b) Hence, or otherwise, find $\overrightarrow{FE}$. **2**

SQA Higher Maths 2016 Non Calc Q7

(a) A and C are the points $(1, 3, -2)$ and $(4, -3, 4)$ respectively.

Point B divides AC in the ratio $1 : 2$.

Find the coordinates of B. **2**

(b) $k\overrightarrow{AC}$ is a vector of magnitude $1$, where $k > 0$.

Determine the value of $k$. **3**

SQA Higher Maths 2016 Non Calc Q11

Vectors $\mathbf{u} = 8\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + t\mathbf{j} - 6\mathbf{k}$ are perpendicular.

Determine the value of $t$. **2**

SQA Higher Maths 2015 Non Calc Q1

(*a*) (i) Show that the points A(–7, –8, 1), T(3, 2, 5) and B(18, 17, 11) are collinear.

(ii) Find the ratio in which T divides AB. **4**

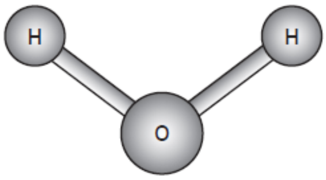
(*b*) The point C lies on the $x$-axis.

If TB and TC are perpendicular, find the coordinates of C. **5**

SQA Higher Maths 2013 Non Calc Q24

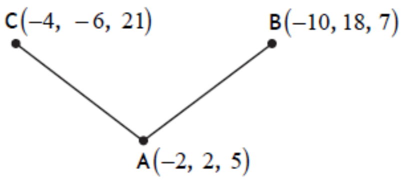
## Calculator Section

The picture shows a model of a water molecule.




Relative to suitable coordinate axes, the oxygen atom is positioned at point A$(-2, 2, 5)$.

The two hydrogen atoms are positioned at points B$(-10, 18, 7)$ and C$(-4, -6, 21)$ as shown in the diagram below.



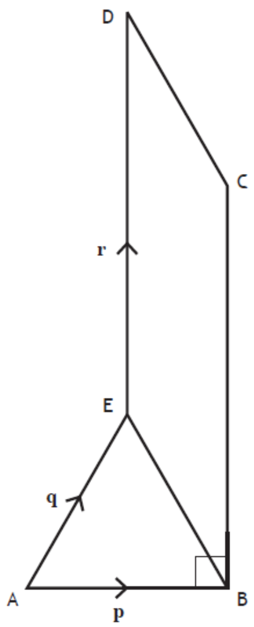

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form. 2

(b) Hence, or otherwise, find the size of angle BAC. 4

SQA Higher Maths 2016 Calc Q5

Vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are represented on the diagram as shown.

- BCDE is a parallelogram
- ABE is an equilateral triangle
- $|\mathbf{p}| = 3$
- Angle ABC $= 90°$




(a) Evaluate $\mathbf{p}.(\mathbf{q}+\mathbf{r})$. 3

(b) Express $\overrightarrow{EC}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. 1

(c) Given that $\overrightarrow{AE}.\overrightarrow{EC} = 9\sqrt{3} - \frac{9}{2}$, find $|\mathbf{r}|$. 3

SQA Higher Maths 2015 Calc Q6

## Practice Assessment Solutions

Practice (1)

(1) $\overrightarrow{FB}$

$= \overrightarrow{FA} + \overrightarrow{AB}$

$= -\overrightarrow{AF} + \overrightarrow{AB}$

$= \begin{pmatrix} 1 \\ 4 \\ -13 \end{pmatrix} + \begin{pmatrix} -4 \\ 8 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 12 \\ -9 \end{pmatrix}$

(2) $\overrightarrow{AB}$

$= \underline{b} - \underline{a}$

$= \begin{pmatrix} 7 \\ -4 \\ 9 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$

$\overrightarrow{BC}$

$= \underline{c} - \underline{b}$

$= \begin{pmatrix} 15 \\ -8 \\ 17 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 9 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ -4 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$

$\overrightarrow{BC} = 2\overrightarrow{AB}$ $\therefore$ $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ and they share a common point B therefore A, B and C are ~~parallel.~~ collinear

(3) 3 : 5 so 8 steps

| P | | | Q | | | | | R |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 6 | 11 | 16 | 21 | 26 | 31 | 36 | 41 |
| 18 | 15 | 12 | 9 | 6 | 3 | 0 | -3 | -6 |

Q (5, 16, 9)

(4) $\overrightarrow{TU} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$

$\overrightarrow{TS} = \begin{pmatrix} -4 \\ -1 \\ 7 \end{pmatrix}$

$|\overrightarrow{TU}| = \sqrt{(-3)^2 + 2^2 + 3^2}$

$= \sqrt{22}$

$\overrightarrow{TS} = \sqrt{66}$

$\overrightarrow{TU}.\overrightarrow{TS} = 12 - 2 + 21$

$= 31$

$\overrightarrow{TU}.\overrightarrow{TS} = |\overrightarrow{TU}||\overrightarrow{TS}|\cos\theta$

$\cos\theta = \dfrac{\overrightarrow{TU}\cdot\overrightarrow{TS}}{|\overrightarrow{TU}||\overrightarrow{TS}|}$ *

$= \dfrac{31}{\sqrt{22}\sqrt{66}}$

$\theta = 35.6°$

* this formula must be stated in terms of S, T and U.

(1) $\vec{PT} = \vec{PS} + \vec{ST}$

$= -\vec{SP} + \vec{ST}$

$= \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 7 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$

(2) $\vec{AB}$ $\vec{BC}$

$= \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}$ $= \begin{pmatrix} 15 \\ -3 \\ -12 \end{pmatrix}$

$\vec{AB} \neq k\vec{BC}$ $\therefore$ $\vec{AB}$ is not parallel to $\vec{BC}$ $\therefore$ A, B and C are not collinear.

(3) R S T

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| -1 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
| 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 |
| 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |

3 : 5

8 steps

S $(5, 1, -9)$

(4) $\vec{PQ} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ $\vec{PR} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ $|\vec{PQ}| = \sqrt{14}$ $|\vec{PR}| = \sqrt{26}$

$\vec{PR} \cdot \vec{PQ} = 5$

$\cos\theta = \dfrac{\vec{PR} \cdot \vec{PQ}}{|\vec{PR}|\,|\vec{PQ}|} = \dfrac{5}{\sqrt{14}\sqrt{26}}$ $\theta = 74.8°$